Let $G = \langle S \rangle$ be a finitely generated group, i.e. $|S| < \infty$, and any $g \in G$ can be written as

$$g = s_1 \cdots s_n, \quad s_i \in S \cup S^{-1}$$

Then we can define length function

$$|g|_S := \min\{n \geq 0 : \exists s_1, \ldots, s_n \in S \cup S^{-1}, \; g = s_1 \cdots s_n\}$$

We can define a metric $d_S : G \times G \to \mathbb{R}_{\geq 0}$

$g, h \in G$, $d_S(g,h) := |g^{-1} h|_S$.

Say for now $|e_G|_S = 0$.

Lemma: (G, d_S) is a metric space.

Visualisation $\to$ Cayley graph $\mathrm{Cay}(G,S)$ is the graph whose vertices are elements of G, and we connect $g, h \in G$ if $\exists s \in S \cup S^{-1}$ s.t. $h = gs$.

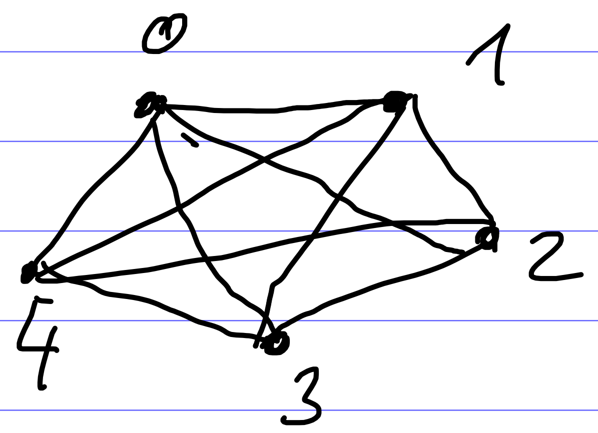
Observations: • As S generates G, $\mathrm{Cay}(G,S)$ is connected.

• There is natural metric on $\mathrm{Cay}(G,S)$, namely the shortest path metric.

Is this metric different from d_S?

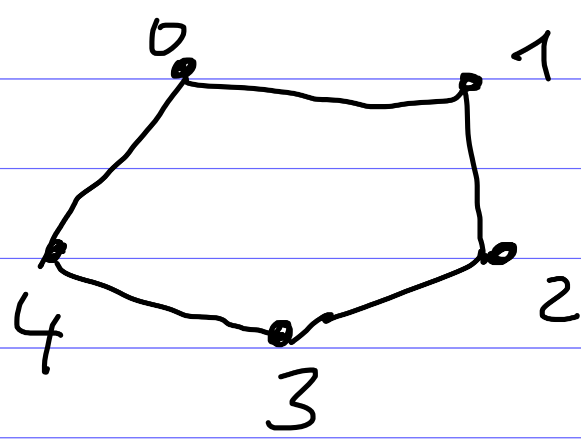
Lemma: (G, d_S), $(Cay(G,S), d_{path})$ are isometric.

Examples: • $G = \mathbb{Z}/5\mathbb{Z}$, $S = G \setminus \{0\}$



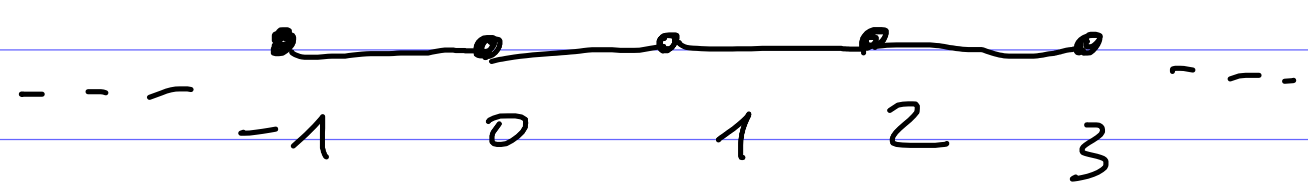
a complete graph

For G, $S' = \{1\}$ $\rightarrow$



a 5-cycle.

• $G = \mathbb{Z}$, $S = \{1\}$:



Because of the previous lemma, we always identify a group with its Cayley graph

Lemma: Let G be a f.g. group, with 2 different finite generating sets $S, R \subset G$. There exist $A, B > 0$ such that

$$A \cdot |g|_R \leq |g|_S \leq B \cdot |g|_R \qquad \forall g \in G.$$

Consequently:

$$A \cdot d_R(g,h) \leq d_S(g,h) \leq B \cdot d_R(g,h)$$

for any $g, h \in G$.

Proof: Let $g \in G$. Write $n := |g|_R$, i.e. $\exists r_1, \dots, r_n \in R \cup R^{-1}$, $g = r_1 \dots r_n$.

As S generates G, we can write

$$r_i := s_{i,1} s_{i,2} \dots s_{i,\ell_i}$$

Then $g = s_{1,1} s_{1,2} \dots s_{1,\ell_1} s_{2,1} \dots s_{2,\ell_2} \dots s_{n,1} \dots s_{n,\ell_n}$ and thus

$$|g|_S \leq \ell_1 + \dots + \ell_n \leq B \cdot n = B \cdot |g|_R$$

where $B := \max\{|r|_S : r \in R\}$. Similarly we have $A \cdot |g|_R \leq |g|_S$,

with $A = (\max\{|s|_R : s \in S\})^{-1}$. $\square$

Definition: Let X, Y be metric spaces. A map $\varphi: X \to Y$ is a quasi-isometry if $\exists C \geq 1, K \geq 0$ s.t:

(i) $\frac{1}{C} d_X(x,y) - K \leq d_Y(f(x), f(y)) \leq C\, d_X(x,y) + K$

for any $x, y \in X$.

(ii) $d_Y(y, f(X)) \leq K \quad \forall y \in Y$, i.e.

$\forall y \in Y, \exists x_y \in X, \ d_Y(y, f(x_y)) \leq K$.

Remark: Being quasi-isometric is an equivalence relation between metric spaces.

Examples: (0). Bounded metric spaces are all quasi-isometric.

(1) The map $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry: any $x \in \mathbb{R}$ is at dist. ≤ 1 from $\lfloor x \rfloor \in \mathbb{Z}$.

More generally, $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$, any $(x_1, \dots, x_n) \in \mathbb{R}^n$ is at distance $\leq \sqrt{n}$ of $(\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$

(2) For any finite group F and any f.g. G, G and $G \times F$ are quasi-isometric.

How to "easily" produce quasi-isometries?

Def: A metric space X is quasi-geodesic if there are $C > 0, K \geq 0$ s.t. any two points $x, y \in X$ are connected by a (C, K)-quasi-geodesic, i.e.

a (C,K)-quasi-isometric embedding

$$\gamma : [0, d(x,y)] \to X$$

s.t. $\gamma(0) = x$, $\gamma(d(x,y)) = y$.

Examples: • Any normed space $(V, \|\cdot\|)$ is geodesic: given $x, y \in V$, we can define $\gamma : [0,1] \to V$, $t \mapsto (1-t)x + ty$.

• Any connected graph with its path metric is quasi-geodesic.

Def: Let G be a group acting on a set X. The action is:

• cobounded if $\exists\, x_0 \in X$ and $R \geq 0$ s.t. any $x \in X$ is at dist. $\leq R$ from $G \cdot x_0$.

• metrically proper if there is $x_0 \in X$ such that $\{g \in G : d_X(x_0, gx_0) \leq R\}$ is finite for any $R \geq 0$.

Theorem (Milnor-Svarc lemma): Let G be a group acting isometrically on X, a quasi-geodesic metric space. If the action is cobounded and metrically proper, then G is finitely generated, and the map

$$\iota_{x_0} : G \longrightarrow X$$
$$g \mapsto g \cdot x_0$$

is a quasi-isometry for any $x_0 \in X$.

Corollary: Let G be a f.g. group. Let $H \leq G$ be a finite-index subgroup. Then H is f.g., and the inclusion map $H \hookrightarrow G$ is a quasi-isometry.

Proof: Let $G = \langle S \rangle$, $|S| < \infty$, and consider $H \curvearrowright (G, d_S)$. It is isometric.

Metric properness: Let $R \geq 0$

$$\{ h \in H : d_S(1_G, h) \leq R \} = H \cap \underbrace{B_{d_S}(1_G, R)}_{\text{finite}}$$

$\Rightarrow$ action is metrically proper.

Co-boundedness: $[G : H] < \infty$, so there are $a_1, \dots, a_r \in G$ s.t.

$$G = Ha_1 \sqcup \dots \sqcup Ha_r$$

Then, given $g \in G$, there is $i \in \{1, \dots, r\}$ s.t. $g \in Ha_i$, and thus

$$d_S(g, H \cdot 1_G) \leq |a_i|_S \leq \max_{1 \leq i \leq r} |a_i|_S$$

so the action is cobounded. Then Milnor-Švarc's lemma gives the result. $\square$

Lemma: A metric space is quasi-geodesic if and if only it is quasi-isometric to a connected graph.

Proof: "$\Leftarrow$": obvious, since connected graphs are quasi-geodesic, and that quasi-geodesicity is preserved by quasi-isometries.

"$\Rightarrow$": let X be a quasi-geodesic metric space, i.e. any $x, y \in X$ are connected by a (C, K)-quasi-geodesic. Let $R := C + K$, and let Y be the graph whose vertices are elements of X, and whose edges connect $x, y \in X$ if $d_X(x,y) \leq R$.

Claim: Y is connected, and the natural $X \to Y$ is a quasi-isometry. $\square$

Our first invariant: the growth type of finitely generated groups.

Def: Let $G = \langle S \rangle$, $|S| < \infty$. The growth function of G w.r.t. S is

$$\gamma_{(G,S)} : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto \gamma_{(G,S)}(n) = |B_{d_S}(1_G, n)|.$$

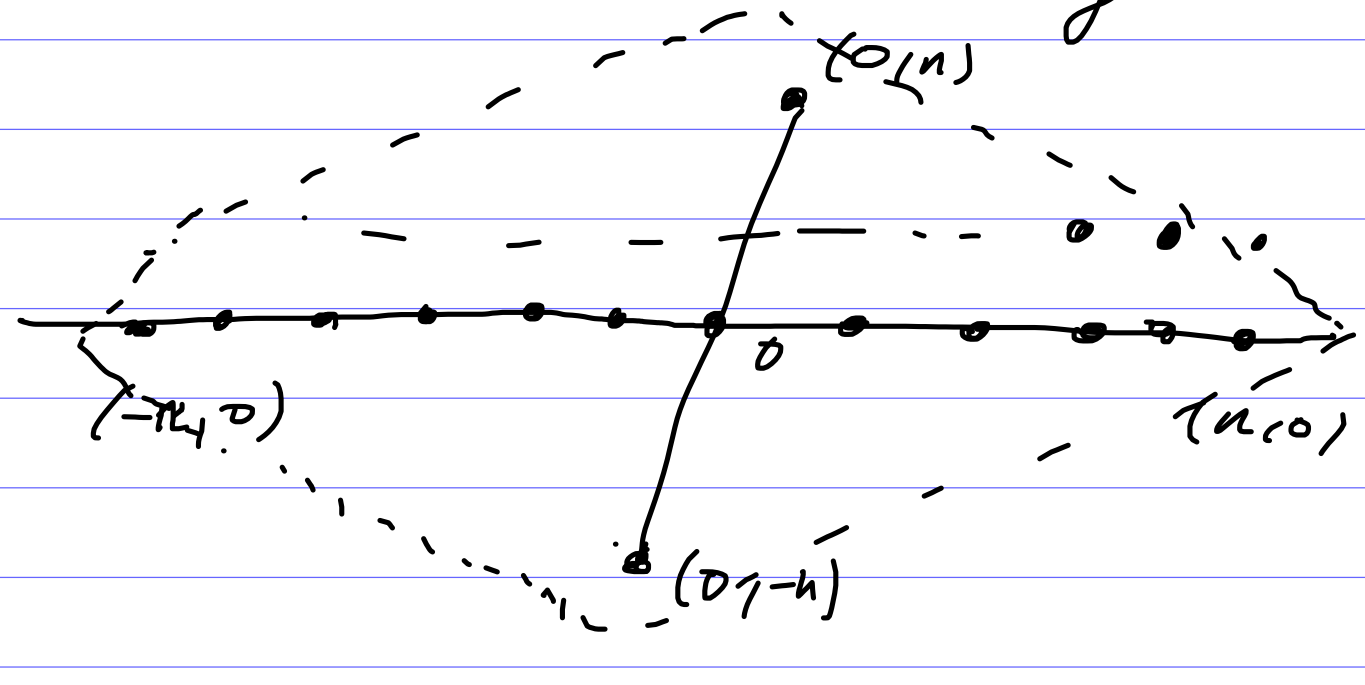
Examples: $G = \mathbb{Z}$, $S = \{\pm 1\}$. Then

$$\gamma_{(G,S)}(n) = |B(0,n)| = |\{-n, \dots, n\}| = 2n+1$$

- $G = \mathbb{Z}^2$, $S = \{\pm(1,0), \pm(0,1)\}$. Then

$$B((0,0), n) = \{(i,j) \in \mathbb{Z}^2 : |i| + |j| \leq n\}$$

i.e.



and then

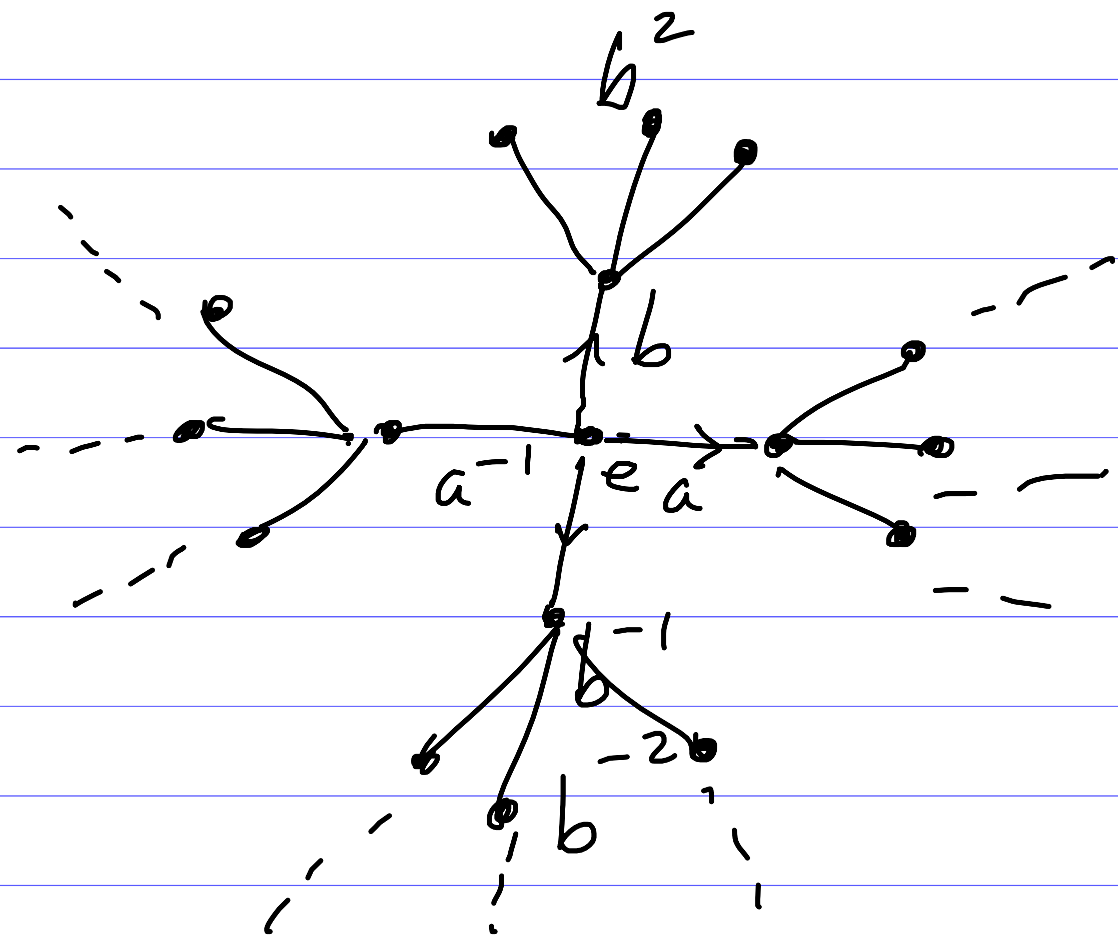
$$\gamma_{(\mathbb{Z}^2,S)}(n) = 2\left(1+3+5+\dots+(2n-1)\right) + (2n+1) = 2n^2+2n+1.$$

If rather $S' = S \cup \{(1,1), (-1,-1), (-1,1), (1,-1)\}$

then $B_{d_{S'}}((0,0), n) = \{-n, \dots, n\}^2$

$$\Rightarrow \gamma_{(G,S')}(n) = (2n+1)^2 = 4n^2+4n+1.$$

- $G = F_2 = \langle a, b \rangle$, $S = \{a^{\pm 1}, b^{\pm 1}\}$.



$$\gamma_{(F_2,S)}(n) = 1 + 4 \cdot \sum_{j=0}^{n-1} 3^j = 1 + 4 \cdot \frac{3^n-1}{2} = 2 \cdot 3^n - 1.$$